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# AGGREGATION AND OTHER REDUCTIONS FOR SEMI-SEPARABLE MARKOV DECISION PROCESSES: THE LINEAR CASE

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## I. Introduction

In a series of recent papers, I show how a semi-separable Markov decision process can be reduced to a problem with n constraints (the number of states) and n lower bounds. If $x \in X$ is a state, and $y \in X$ is a feasible decision, then a process is "semi-separable" (Mendelssohn 1978a)

$$P^y_{x,j} = P^y_j \quad \text{for all } x, j \in X; \text{ and all } y$$

where $P^y_{x,j}$ is the transition probability to state j. Let $f_0, f_1, \ldots, f_n$ be the optimal values of the Markov process, and let $v_0 = f_0$; $v_1 = f_1 - f_0, \ldots, v_n = f_n - f_{n-1}$. Then I show the LP that solves the problem can be formulated as:

$$\begin{aligned} \min \sum_{x=0}^{n} (n+1-x)v_x & \\ \text{s.t.} \quad Av \geq b & \qquad (1.1) \\ v_i \geq b^*_j & \end{aligned}$$

where the j+1 element of b is the return from remaining in state j, $b^*_j$ are simply determined lower bounds, and the $j^{th}$ row of A is:

$$\left(1-\alpha,\ 1-\alpha \sum_{i=1}^{n} P^j_i,\ ---,\ 1-\alpha\sum_{i=j}^{n} P^j_i,\ -\alpha\sum_{i=j+1}^{n} P^j_i,\ \ldots,\ -\alpha P^j_n\right).$$

(see Mendelssohn 1978a for details). In Mendelssohn (1978b), I use the structure of this reduced LP to show how to easily recover an optimal policy from a solution to (1.1), and also to describe qualitative properties of an optimal policy for a wide class of semi-separable processes.

In this paper, I focus on a special case; where the decision set $Y(x)$ for each $x \varepsilon X$ is defined as:

$$Y(x) \equiv \{y : 0 \leq y \leq x\} \tag{1.2}$$

and the return is linear, that is:

$$G(x, y) = h \cdot (x-y); \; h \geq 0 \tag{1.3}$$

This is an example of a separable problem considered by Denardo (1968). However, (1.1) is a smaller problem than the equivalent problem in Denardo; and can be used to obtain even greater reductions in problem size.

In Mendelssohn (1978b), I show that a separable problem satisfying (1.2) and (1.3) always has a base stock policy optimal, if an optimal policy exists. Also, in the linear case, each $b_j^*$ is identically h. the lower bounds can be removed by substituting $v_i' = v_i - h$ for $i = 1, \ldots, n$. Before stating theorem 1, a lemma is needed.

*Lemma 1.1* *Let x be a nonnegative random variable defined on* [0, b], *where* $b \leq \infty$ , *and* $F(x)$ *its cdf.* *Then:*

$$E(x) = \int_0^b [1 - F(x)] \, dx$$

*Proof*. See Karlin and Taylor (1975), p. 38.

□

<u>Theorem 1.1</u> <u>Assumption (1.2) and (1.3) imply the LP given (1.1) is equivalent to:</u>

$$\begin{aligned} &\text{minimize} \quad \sum_{x=0}^{n} v'_x \\ &\text{s.t.} \qquad Av \geq b \end{aligned} \qquad (1.4)$$

<u>where $b^i = \alpha E(s[i]) - (i+1)h$; and $s[i]$ is the transition function from state $i$.</u>

<u>Proof</u>. By the substitution, $v_i = v'_i + h$. At i, this gives $A^i v$ and terms multiplies by h.

$$h \cdot \left( (i+1) - \alpha\left(1 + \left(1-p_0^i\right) + \left(1-p_0^i - p_1^i\right) + \text{---} + \left(1-p_0^i - p_1^i \text{ --- } -p_n^i\right)\right)\right)$$

The inner terms are just $(i+1) - \alpha \sum_{i=0}^{n} \left(1-F(i)\right) = (i+1) - \alpha E(s[i])$ by lemma 1.1.

$\square$

It is interesting to note that the $(i+1)^{st}$ RHS is greater than the $i^{th}$, if and only if:

$$E(s[i+1]) - E(s[i]) \geq \frac{h}{\alpha} \qquad (1.5)$$

In a continuous state version, this is equivalent to:

$$\frac{d}{di}\left(Es(i)\right) \geq \frac{1}{\alpha} \qquad (1.6)$$

Equality in (1.6) is the two-period base stock size. In each row i of A, $v^i$ is added to the row before. The algorithm checks that this "weighted derivative" is greater or less than the change in the two-period result. A related result can be found in Mendelssohn and Sobel (1977).

## II. Reducing the Size of the LP

For many problems, even (1.4) will be a very large LP. Luckily, it is possible to greatly reduce (1.4). It is convenient for this discussion to let $b \equiv 0$, and retain the lower bounds $v_i \geq h$.

Often it is known a priori that the base stock size lies in the interval $(j^L, j^u)$. This suggests it may be possible to aggregate all the variables below $j^L$ and all the variables above $j^u$. Theorem 2.1 states that the variables above $j^u$ may be aggregated without loss of optimality; the variables below $j^L$ may be aggregated but possibly with loss of optimality. However, the base stock size will still be in $(j^L, j^u)$.

Aggregate variables by letting:

$$c^u = \frac{1}{n-j^u+1}\left(\sum_{i=j^u}^{n} (n+1-i)\right) ; \quad c^L = \sum_{i=0}^{j^L} w^i(n+1-i)$$

where $$w^k = \left(\left(1-\alpha\sum_{i=k}^{n} P_i^{j^L}\right) \Big/ \sum_{i=0}^{j^L}\left(1-\alpha\sum_{k=i}^{n} P_k^{j^L}\right)\right)$$

and let $$q_i^u = \frac{1}{n+1}\left(\sum_{k=j^u}^{n} a_k^i\right) ; \quad q_i^L = \sum_{k=0}^{j^L}\left(w^k a_k^i\right) : i = j^L, j^L+1, \ldots, j^u$$

where $a_k^i$ is the (i, k) element of A. Let $\bar{v}^T = (v^L, v_{j^L+1}, \ldots, v_{j^u-1}, v^u)$, and $\bar{A}$ be A with rows 1 through $j^L$, $j^u+1$ through n+1 eliminated; columns 1 through $j^L$, $j^u+1$ through n+1 eliminated, and the first and last columns of $\bar{A}$ are $q^L$ and $q^u$, where $q^{L^T} = \left(q^L_{j^L}, q^L_{j^L+1}, \ldots, q^L_{j^u}\right)$ and $q^{u^T} = \left(q^u_{j^L}, q^u_{j^L+1}, \ldots, q^u_{j^u}\right)$.

<u>Theorem 2.1</u> <u>The reduced problem:</u>

$$\min \sum_{i=j^L+1}^{j^u-1} (n+1-i)v_i + c^u v^u_u + c^L v^L \tag{1.7}$$

$$\text{s.t.} \quad \bar{A}\bar{v} \geq 0$$

$$v_i \geq h,\ i = j^L+1, \ldots, j^u-1;\ v^u \geq (n-j^u+1)h;\ v^L \geq (j^L+1)h$$

<u>has the following properties:</u>

(i) <u>The base stock size</u> $\bar{j}$ <u>lies in</u> $(j^L, j^u)$.

(ii) <u>There is no loss of optimality from aggregating</u> $v_{j^u}, v_{j^u+1}, \ldots, v_n$

(iii) <u>The loss of optimality from aggregating</u> $v_0, v_1 \ldots, v_{j^L}$ <u>is at most</u>

$$\left[\max_{i = 0,1,\ldots,j^L}\left\{(n+1-i) - \bar{u}A^i\right\}\right]^+ \frac{\alpha}{1-\alpha} hn,$$

where $\bar{u}$ are the dual variables of the reduced problem.

<u>Proof</u>. A heuristic proof is offered for aggregating the upper variables, since a rigorous proof can be derived as a special case of the lower variables, which is more difficult. Since $j^* \in (j^L, j^u)$, each $v_i$, $i \geq j^u$,

must be identically h. Also, rows $j^u+1$ through n+1 cannot be tight, hence they can be eliminated. Averaging the transition is justified in that the new process, if it goes to a state greater than $j^u$, immediately goes at least to $j^u$(perhaps lower). These are the $q_i^u$'s.

For the lower variables, it is first necessary to show that $j^L$ will not come out as the solution to (1.7). Suppose it does. The proof is to show that this implies $j^* \notin (j^L, j^u)$ in the original problem. Disaggregate by letting

$$v_i = w^i v^L \qquad i = 0, \ldots, j^L$$

$$v_i = \frac{1}{n-j^u+1} v^u \qquad i = j^u, \ldots, n+1$$

Then from proposition (1) in Zipkin the disaggregate solution is feasible in (1.1), and equality must hold in row $j^L+1$, due to how the weights $w^i$ were chosen.

Since $v_{j^L+1}, \ldots, v_n$ are at their lower bound and the coefficients on them are negative in that row, it is clear that an improved solution can only be obtained by decreasing $v_0, v_1, \ldots, v_{j^L}$. This implies $v_{j^*-1}$ is at its lower bound, which contradicts the optimality of $j^*$.

Part (iii) is equation (4) in Zipkin (1977).

□

Theorem 2.2 suggests a method for finding an optimal policy. Aggregate together all variables with an index less than $j^L$, greater than $j^u$, and choose a partition on the remaining states, and aggregate up to each of those points. The following theorem shows that if the base stock size found from this problem is at the variable representing the $i^{th}$ set of aggregated states, then the true base stock size must lie in that partition.

_Theorem 2.2_ _Assume_ $j^* \varepsilon (j^L, j^u)$. _Divide_ $(j^L, j^u)$ _into intervals_ $\left(j^L, j^1\right], \left(j^1, j^2\right], \ldots, \left(j^k, j^u\right)$. _Aggregate variables_ $v^0, \ldots, v_{j^L}$ _and_ $v_{j^u}, \ldots, v_n$ _as in theorem 2.1. Aggregate variables_ $v_{j^i}, v_{j^{i+1}}, \ldots, v_{j^{i}-1}$ _similarly to_ $v_0, \ldots, v_{j^L}$, _where the weights_ $w^i$ _are chosen from row_ $j^i$ _as in theorem 2.1. Then_

(i) _An optimal base stock size for the aggregated problem lies in_ $(j^L, j^u)$.

(ii) _If_ $j^i$ _is an optimal base stock size for the aggregate problem,_ $j^* \varepsilon \left(j^i, j^{i+1}\right)$.

(iii) _The error for the aggregate problem is at most_

$$h\,\frac{\alpha n}{1-\alpha}\sum_{i=1}^{K}\left\{\left[\max_{j=j^i,\ \ldots,\ j^{i+1}-1}\left(n+1-(i+j)-\bar{u}A^{(i+j)}\right)\right]^+\right\}$$

$$+\ h\,\frac{\alpha n}{1-\alpha}\left(\left[\max_{i=0,\ \ldots,\ j^L}\left(n+1-i-\bar{u}A^i\right)\right|^+\right)$$

where $\bar{u}$ is the vector of dual variables of the aggregated problem and $A^i$ is the $i^{th}$ column of A.

_Proof_. Part (iii) is again equation (4) in Zipkin (1977). If part (i) isn't true, then either the row equivalent to staying in $j^L$ of $j^u$ must be an equality. The contrapositive proof in this case is the same as in theorem 2.1.

Suppose $j^i$ is the optimal base size for the aggregate problem. Assume $j^* < j^i$, then a similar contrapositive argument yields that $j^*$ is not optimal in the original problem. If $j^* > j^{i+1}$, them set all the $v_i$'s, $i \geq j+1$ equal to h. Since this row is an inequality, it must be an equality in the original row. From the form of the disaggregated row, this can only come from changing $v_0, v_1, \ldots, v_{j^{i+1}}$, so that $v_{j^*-1} \equiv h$, which contradicts the optimality of $j^*$.

□

An efficient search technique can now be devised. Aggregate to $j^L$ and from $j^u$ to n+1, and some small partition of $(j^L, j^u)$. If the partition is 10 more variables, this implies initially an LP that is (12x12)! Given the solution to this problem, reaggregate up to $j^i$ and down to $j^{i+1}$, where $j^i$ is the base stock size from the aggregate problem. Partition this interval as before, and repeat. This algorithm involves solving a series of extremely small LP's. The a posterior error as given by (iii) can be used as a stopping rule for when it is no longer worth going further.

## III. <u>Finding the Interval $(j^L, j^u)$</u>

Consider the problem:

$$f(x) = \max_{0 \leq y \leq x} \left\{ h \cdot (x-y) + \alpha E\Big(f(s[y, D])\Big) \right\}$$

where D is a random variable, and it is assumed that $f(\cdot)$ and $s[\cdot, D]$ are differentiable (perhaps one-sided) and that $f(\cdot)$ is unimodal.

Assumptions that are sufficient for these conditions to be valid are given in Mendelssohn and Sobel (1977).

It is readily seen from that paper or the LP (1.1) that $f'(x) \geq h$. In fact, $f'(x) \equiv h$ if $x \geq y^*$, where $y^*$ is an optimal base stock size. Therefore, if:

$$h \leq \alpha E\left\{h \cdot s^{[1]}[y, D]\right\} = \alpha h E\left\{s^{[1]}[y, D]\right\}$$

the base stock size must be no greater than y. Let $y^*$ be the point where $E(s^{[1]}[y, D]) = \frac{1}{\alpha}$. Then the base stock size for the infinite horizon problem can be no greater than $y^*$. Let $j^u \equiv y^*$. If $s[y, D] = Ds[y]$, $Pr\{D \geq 0\} = 1$, and $s[1]$ is continuous, it is easily seen that if $\bar{y}^*$ is the solution to $s'[y] = \frac{1}{\alpha}$, then the infinite horizon problem base stock size is less than $\bar{y}^*$ if $E(D) > 1$, is greater than $\bar{y}^*$ if $E(D) < 1$, and is less than or equal to $\bar{y}^*$ if $E(D) \equiv 1$.

Thus, in comparing the stochastic and deterministic problems, if $\bar{y}^*$ is the deterministic solution, the stochastic solution is more conservative if and only if $E(s[\bar{y}^*, D]) - \bar{y}^*$ is a supermartingale. Using the deterministic solution as the base stock size is more conservative if $E(s[\bar{y}^*, D]) - \bar{y}^*$ is a submartingale. Unless $E(s[\bar{y}^*, D]) \gg \bar{y}^*$, it probably will not be worth it to calculate the stochastic optimum. Zipkin's (1977) results can be used to determine bounds on using this policy. However, particularly in situations where very low stock levels are highly undesirable, such as in managing renewable resources, using the deterministic solution as the base stock size to lower the risk of depletion will probably outweigh the loss due to this bound. I conjecture that a similar result is valid when linear returns are combined with linear smoothing costs.

In a future paper, these results will be used to analyze several published models relating to managing stochastically varying fisheries.

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